Math 250

5.5 Integration by Substitution - 2nd

Objectives

1) Use a change of variables (substitution) to evaluate a definite integral

2) Evaluate a definite integral involving an even or odd function

Math 250

Definite Integrals by u-substitution.

e.g. 1 $\displaystyle\int_{-1}^{1} x^2 (x^3+2)^{1/2}\, dx$

* CAUTION * Do not plug x-values into u-variables or u-values into x-variables. Must be u into u or x into x.

option 1: use x-values to calculate final answer

option 2: use u-values to calculate final answer.

option 1: $u = x^3+2$

$du = 3x^2\, dx$

$$= \frac{1}{3}\int_{x_1=-1}^{x_2=1} (x^3+2)^{1/2} \cdot 3x^2\, dx$$

$$= \frac{1}{3}\int_{u_1}^{u_2} u^{1/2}\, du$$

(we didnt figure out what u_1 and u_2 are)

$$= \frac{1}{3}\cdot\frac{2}{3} u^{3/2} \Big|_{u_1}^{u_2}$$

$$= \frac{2}{9}(x^3+2)^{3/2} \Big|_{x_1=-1}^{x_2=1}$$

$$= \frac{2}{9}\left[(1^3+2)^{3/2} - ((-1)^3+2)^{3/2}\right]$$

$= \frac{2}{9}\left(3^{3/2} - 1^{3/2}\right)$

$= \boxed{\frac{2}{9}\left(3^{3/2} - 1\right)}$

$= \boxed{\frac{2}{9}\left(3\sqrt{3} - 1\right)} = \boxed{\frac{2}{3}\sqrt{3} - \frac{2}{9}}$

<u>option 2:</u> avoids $(\text{mess})^{3/2}$ in evaluate.

$$\int_{x_1=-1}^{x_2=1} x^2 (x^3+2)^{1/2}\, dx$$

$u = x^3 + 2$

$du = 3x^2\, dx$

$u = x^3 + 2$ means

$u_1 = (x_1)^3 + 2$ and $u_2 = (x_2)^3 + 2$

$u_1 = (-1)^3 + 2$ $\quad u_2 = 1^3 + 2$

$u_1 = 1$ $\quad u_2 = 3$

$$= \frac{1}{3}\int_{u_1=1}^{u_2=3} u^{1/2}\, du$$

$$= \frac{1}{3}\cdot\frac{2}{3} u^{3/2} \Big|_{u_1=1}^{u_2=3}$$

$$= \frac{2}{9}\left(3^{3/2} - 1^{3/2}\right)$$

$$= \boxed{\frac{2}{9}\left(3^{3/2} - 1\right)}$$

Math 250 Definite Integrals by Substitution

Either of the two approaches is valid — but don't get confused and do half one approach and half the other.

- Every evaluation at end is a u-value

OR

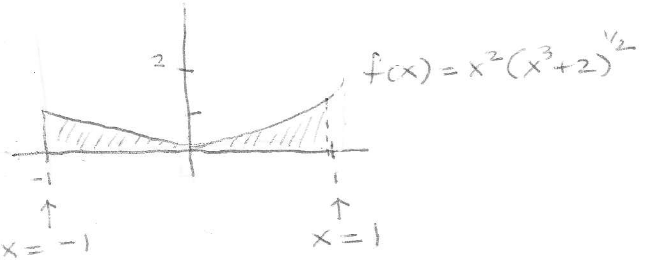
- Every evaluation at end is an x-value

Integrate — same example with graphs in x & u

① $\displaystyle\int_{-1}^{1} x^2(x^3+2)^{\frac{1}{2}}\,dx$

Notice that these limits of integration refer to values of x:

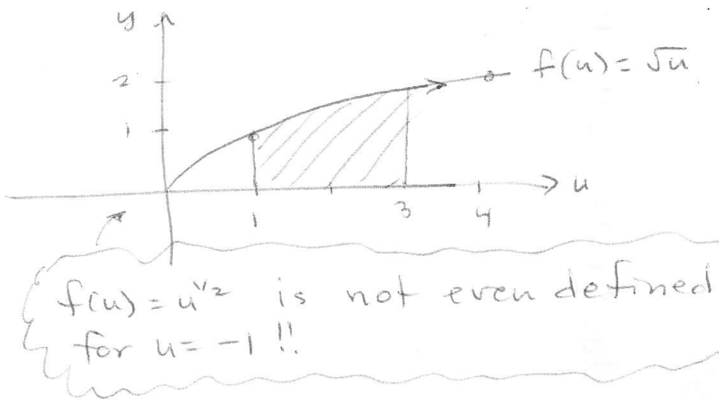
$$= \frac{1}{3}\int_{-1}^{1} (x^3+2)^{\frac{1}{2}} \cdot 3x^2\,dx$$



But if we change variables, we get an entirely different graph:

$$\left.\begin{aligned} u &= x^3+2 \\ du &= 3x^2\,dx \end{aligned}\right\}$$

$$= \frac{1}{3}\int_{u=?}^{u=?} u^{1/2}\,du$$



We must find new limits of integration

$u(-1) = (-1)^3+2 = -1+2 = 1$ ← $x=-1$ was lower, so $u=1$ is lower

$u(1) = (1)^3+2 = 1+2 = 3$ ← $x=1$ was upper, so $u=3$ is upper

$$= \frac{1}{3}\int_{u=1}^{u=3} u^{1/2}\,du$$

$$= \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=3} \quad \leftarrow \text{integrate}$$

$$= \frac{1}{3}\cdot\frac{2}{3}\left(3^{3/2} - 1^{3/2}\right) \quad \leftarrow \text{evaluate using } u\text{-limits}$$

$$= \boxed{\frac{2}{9}\left[(\sqrt{3})^3 - 1\right]} \quad \text{exact.}$$

$\approx 0.932478316 2$

To check ↑, calculate

GC check: (original integral)

$Y_1 = x^2\sqrt{x^3+2}$

$\text{fnInt}(Y_1, X, -1, 1) = .9324783161$

GC check: (subst integral)

$Y_2 = \sqrt{u} = \sqrt{x}$

$Y_3 *\ \text{fnInt}(Y_2, X, 1, 3) = .9324783162$

The areas of the plane regions of the two graphs are equal! ☺

Math 250

Note: Be very careful with notation:

$\int_2^3 f(x)\,dx$ } means you have not yet found the antiderivative

$F(x)\Big|_{x=2}^{x=3}$ or $F(x)\Big|_2^3$ } means you have found the antiderivative and now need to evaluate.

CAUTION: Do not substitute u-values for limits of integration into x variables.

Do not substitute x-values for limits of integration into u variables.

i.e. do not mix u and x variables!!

$$\int_{u=u_1}^{u=u_2} f(u)\,du = F(u_2) - F(u_1) \qquad F'(u) = f(u) \quad \{\text{not } F'(x) = g(x)\}$$

but

$$\int_{x=x_1}^{x=x_2} g(x)\,dx = G(x_2) - G(x_1) \qquad G'(x) = g(x) \quad \{\text{not } G'(u) = f(u)\}$$

Not $\int_{u_1}^{u_2} f(u)\,du = G(u_2) - G(u_1)$ OR $\int_{x_1}^{x_2} g(x)\,dx = F(x_2) - F(x_1)$

Math 250

(2) $\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx$

$u = 1+2x^2$

$du = 4x\,dx$

$u(0) = 1+2(0)^2 = 1$ (lower)

$u(2) = 1+2(2)^2 = 9$ (upper)

$= \frac{1}{4}\int_0^2 \frac{4x\,dx}{\sqrt{1+2x^2}}$ ← introduce $\frac{1}{4}$ and 4

$= \frac{1}{4}\int_{u=1}^{u=9} \frac{du}{\sqrt{u}}$ ← subst for u

$= \frac{1}{4}\int_1^9 u^{-1/2}\,du$ ← rewrite exp

$= \left[\frac{1}{4}\cdot\left[2u^{1/2}\right]\right]\Big|_{u=1}^{9}$ ← integrate

$= \frac{1}{2}(\sqrt{9}-\sqrt{1})$ ← evaluate @ u-limits

$= \frac{1}{2}(3-1)$

$= \boxed{1}$

(3) $\int_1^9 \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx$

$u = 1+\sqrt{x}$

$du = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$u(1) = 1+\sqrt{1} = 2$ lower

$u(9) = 1+\sqrt{9} = 4$ upper

$= 2\int_1^9 \frac{1}{2\sqrt{x}\,(1+\sqrt{x})^2}\,dx$

$= 2\int_2^4 \frac{du}{u^2}$

$= 2\int_2^4 u^{-2}\,du$

$= 2\left[(-1)u^{-1}\right]\Big|_{u=2}^{u=4}$

$= -2\left[\frac{1}{u}\right]_{u=2}^{u=4}$

$= -2\left(\frac{1}{4}-\frac{1}{2}\right)$

$= -2\left(-\frac{1}{4}\right)$

$= \boxed{\frac{1}{2}}$

(4) $\int_0^2 x\sqrt[3]{4+x^2}\,dx$

$u = 4+x^2$ $\quad u(0) = 4+0^2 = 4$

$du = 2x\,dx$ $\quad u(2) = 4+2^2 = 8$

$= \frac{1}{2}\int_0^2 \sqrt[3]{4+x^2}\cdot 2x\,dx$

$= \frac{1}{2}\int_4^8 u^{1/3}\,du$

$= \frac{1}{2}\cdot\frac{3}{4}\left[u^{4/3}\right]\Big|_{u=4}^{u=8}$

$= \frac{3}{8}\left(8^{4/3} - 4^{4/3}\right)$

$= \frac{3}{8}\left(2^4 - \left(\sqrt[3]{2^2}\right)^4\right)$

$= \frac{3}{8}\left(16 - \left(\sqrt[3]{2}\right)^8\right)$

$= \frac{3}{8}\left(16 - 2\cdot 2\cdot\sqrt[3]{4}\right)$

$= \frac{12}{8}\left(4 - \sqrt[3]{4}\right)$

$= \frac{3}{2}\left(4-\sqrt[3]{4}\right) = \boxed{6 - \frac{3}{2}\sqrt[3]{4}}$

Math 250

Sometimes the substitution takes more work.

(5) $\int x\sqrt{4x+1}\,dx$

$u = 4x+1$

$du = 4\,dx$

But what to do with the extra x?

* Solve $u = 4x+1$ for x to get x in terms of u *

$u = 4x+1$

$u - 1 = 4x$

$\frac{1}{4}u - \frac{1}{4} = x$

$= \frac{1}{4}\int x\sqrt{4x+1}\cdot 4\,dx$

$= \frac{1}{4}\int \left(\frac{1}{4}u - \frac{1}{4}\right)\cdot\sqrt{u}\cdot du$ ← subst $x = \frac{1}{4}u - \frac{1}{4}$, $\sqrt{4x+1} = \sqrt{u}$, $4\,dx = du$

$= \frac{1}{4}\cdot\frac{1}{4}\int (u-1)\cdot u^{1/2}\,du$ ← factor out $\frac{1}{4}$

$= \frac{1}{16}\int \left(u^{3/2} - u^{1/2}\right)du$ ← dist

$= \frac{1}{16}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$ ← integrate

$= \frac{1}{16}\cdot\frac{2}{15}\cdot u^{3/2}\left[3u - 5\right] + C$ ← factor out
- GCF numerators
- LCD denoms
- lowest power of u.

$= \frac{1}{120}(4x+1)^{3/2}\left[3(4x+1) - 5\right) + C$

$= \frac{1}{120}(4x+1)^{3/2}(12x + 3 - 5) + C$

$= \frac{1}{120}(4x+1)^{3/2}(12x - 2) + C$

$= \boxed{\frac{1}{60}(4x+1)^{3/2}(6x-1) + C}$

Math 250

(6) $\int (x+1)\sqrt{2-x}\, dx$

$$\left.\begin{aligned} u &= 2-x \\ du &= -\cancel{x}\, dx \end{aligned}\right\} \quad \text{neg 1 needed.}$$

$$\left.\begin{aligned} u &= 2-x \\ u+x &= 2 \\ x &= 2-u \end{aligned}\right\}$$

$= -\int (x+1)\sqrt{2-x} \cdot (-1)\, dx$

$= -\int (2-u+1)\sqrt{u} \cdot du$ ← subst $u = 2-x$, $du = -dx$, $x = 2-u$

$= -\int (3-u)\, u^{1/2}\, du$ ← simplify

$= -\int \left(3u^{1/2} - u^{3/2}\right) du$ ← distribute

$= -\left[3 \cdot \frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2}\right] + C$ ← integrate

$= -\left[2u^{3/2} - \frac{2}{5} u^{5/2}\right] + C$ ← simplify

$= -\frac{2}{5} u^{3/2}\left[5 - u\right] + C$ ← factor out $\left\{\begin{array}{l} \text{GCF numerators} \\ \text{LCD denominators} \\ \text{least power of } u \end{array}\right.$

$= -\frac{2}{5}(2-x)^{3/2}\left(5-(2-x)\right) + C$ ← subst $u = 2-x$ back

$= \frac{-2}{5}(2-x)^{3/2}(5-2+x) + C$ ← dist

$= \boxed{\frac{-2}{5}(2-x)^{3/2}(x+3) + C}$ ← simplify

Math 250

(7) $\displaystyle\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

$u = 2x-1$

$du = 2\,dx$

$u(1) = 2(1)-1 = 1$ (lower)

$u(5) = 2(5)-1 = 9$ (upper)

$2x-1 = u$

$2x = u+1$

$x = \frac{1}{2}(u+1)$

subst $\left\{\begin{array}{l} u = 2x-1 \\ du = 2\,dx \\ x = \frac{1}{2}(u+1) \\ u(1) = 1 \\ u(5) = 9 \end{array}\right.$

$= \frac{1}{2}\displaystyle\int_1^5 \frac{x}{\sqrt{2x-1}} \cdot 2\,dx$

$= \frac{1}{2}\displaystyle\int_1^9 \frac{\frac{1}{2}(u+1)}{\sqrt{u}} \cdot du$

$= \frac{1}{4}\displaystyle\int_1^9 \frac{u+1}{u^{1/2}}\,du$ ← factor out $\frac{1}{2}$

$= \frac{1}{4}\displaystyle\int_1^9 \left(\frac{u}{u^{1/2}} + \frac{1}{u^{1/2}}\right) du$ ← separate numerator

$= \frac{1}{4}\displaystyle\int_1^9 \left(u^{1/2} + u^{-1/2}\right) du$ ← subtract exp

$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2u^{\frac{1}{2}}\right]\Big|_{u=1}^{u=9}$ ← integrate

$= \frac{1}{4}\left\{\left(\frac{2}{3}(9)^{3/2} + 2\cdot(9)^{1/2}\right) - \left(\frac{2}{3}\cdot(1)^{3/2} + 2\cdot(1)^{\frac{1}{2}}\right)\right\}$ ← evaluate

$= \frac{1}{4}\left\{\left(\frac{2}{3}(3^3) + 2\cdot 3\right) - \left(\frac{2}{3} + 2\right)\right\}$

$= \frac{1}{4}\left\{18 + 6 - \frac{2}{3} - 2\right\}$

$= \boxed{\frac{16}{3}}$

Math 250

(8) Solve the DE

$$\frac{dy}{dx} = 4x + \frac{9x^2}{(3x^3+1)^{(3/2)}}$$

$$y(0) = 2.$$

$$y(x) = \int \left(4x + \frac{9x^2}{(3x^3+1)^{3/2}}\right) dx$$

$$= 4\int x\,dx \qquad \int \frac{9x^2}{(3x^3+1)^{3/2}}\,dx$$

$u = 3x^3+1$

$du = 9x^2\,dx$

$$= 4\cdot \frac{1}{2}x^2 + c \qquad \int \frac{1}{u^{3/2}}\,du$$

Do integrals separately to avoid mismatch of x's and u's

$$= 2x^2 + c \qquad \int u^{-3/2}\,du$$

$$= 2x^2 + c \qquad \left(\frac{-2}{1}\right)u^{-1/2} + c$$

$$y(x) = 2x^2 + \frac{-2}{\sqrt{3x^3+1}} + c$$

$$y(0) = 2(0)^2 + \frac{-2}{\sqrt{3(0)^3+1}} + c = 2$$

$$0 + \frac{-2}{1} + c = 2$$

$$c = 4$$

$$\boxed{y(x) = 2x^2 - \frac{2}{\sqrt{3x^3+1}} + 4}$$

or

$$\boxed{y(x) = 2\left[x^2 - \frac{1}{\sqrt{3x^3+1}} + 2\right]}$$

Math 250

(9) $\displaystyle\int_{\pi/12}^{\pi/4} \csc 2x \cdot \cot 2x \, dx$

$u = 2x$

$du = 2\,dx$

$u(\pi/12) = 2(\pi/12) = \pi/6$ lower

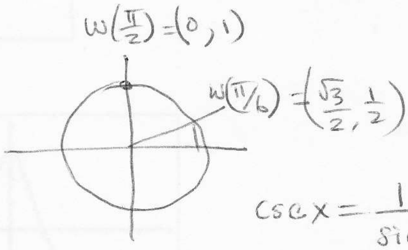
$u(\pi/4) = 2(\pi/4) = \pi/2$ upper

$$= \frac{1}{2}\int_{\pi/12}^{\pi/4} \csc 2x \cdot \cot 2x \cdot 2\,dx$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/2} \csc u \cot u \, du$$

$$= -\frac{1}{2}\left[\int_{\pi/6}^{\pi/2} -\csc u \cot u \, du\right]$$

$$= \frac{-1}{2}\left[\csc u\right]\Big|_{u=\pi/6}^{u=\pi/2}$$



$$= \frac{-1}{2}\left(\csc\frac{\pi}{2} - \csc\frac{\pi}{6}\right)$$

$\csc x = \dfrac{1}{\sin x}$

$$= \frac{-1}{2}(1 - 2)$$

$$= \frac{-1}{2}(-1)$$

$$= \boxed{\frac{1}{2}}$$

$\csc 2x = \dfrac{1}{\sin 2x}$ $\qquad \cot 2x = \dfrac{\cos 2x}{\sin 2x}$

GC check: $y_1 = \csc(2x) \cdot \cot(2x) = \cos(2x) / [\sin(2x)]^2$

fnInt$(Y_1, X, \pi/12, \pi/4) = .5$ ✓

(10)

Evaluate $\displaystyle\int_0^{\pi/2} \cos^2\theta \, d\theta$

$$\left.\begin{aligned} &\cos(2\theta) = 1 - 2\cos^2\theta \\ &\cos^2\theta = \frac{\cos 2\theta - 1}{-2} = \frac{1-\cos 2\theta}{2} \end{aligned}\right\} \text{trig identity}$$

$$\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(1-\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1-\cos 2\theta \, d\theta$$

$u = 2\theta$
$du = 2\, d\theta$
$u_1 = 2(0) = 0$
$u_2 = 2(\frac{\pi}{2}) = \pi$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{\pi} 1-\cos u \, du$$

$$= \frac{1}{4}\int_0^{\pi} 1-\cos u \, du$$

$$= \frac{1}{4}\int_0^{\pi} 1\, du - \frac{1}{4}\int_0^{\pi}\cos u \, du$$

$$= \frac{1}{4} u \Big|_0^{\pi} - \frac{1}{4}\sin u \Big|_0^{\pi}$$

$$= \frac{1}{4}(\pi) - \frac{1}{4}(0) - \left[\frac{1}{4}\sin\pi - \frac{1}{4}\sin 0\right]$$

$$= \frac{\pi}{4} - 0$$

$$= \boxed{\frac{\pi}{4}}$$